Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 68 (2004) 631–646

CONSTRUCTION OF THE ATTAINABILITY SET OF A BROCKETT INTEGRATOR†

S. A. VDOVIN, A. M. TARAS'YEV and V. N. USHAKOV

Ekaterinburg

e-mail: ushak@imm.uran.ru

(*Received* 8 *April* 2004)

Methods of optimal control theory [1, 2] are used to solve the problem of constructing attainability sets for a non-linear dynamical system known as the non-holonomic Brockett integrator [3]. It is proved that the boundaries of attainability sets are characterized by points of optimal trajectories constructed for a control problem whose integral performance index defines the area of the figure bounded by a trajectory of motion of the controlled system. The problem is to maximize that area. This formulation is similar to that of Dido's problem in the calculus of variations, namely, to construct a figure of maximum area for a given perimeter. An algorithm is proposed for constructing optimal trajectories and their properties are investigated. The algorithm is based on results obtained by solving a special case of the optimal control problem with a closed trajectory of motion.The necessary optimality conditions of Pontryagin's Maximum Principle are verified for the optimal trajectories constructed. Analytical formulae are derived for the value function of the control problem and the necessary and sufficient conditions for optimally are verified using Subbotin's minimax inequalities for Hamilton–Jacobi equations.

The Brockett integrator [3] is one of the first classical examples of a system for which solution of the control problem requires the introduction of a non-linear (discontinuous) control law. After a suitable change of variables, this system describes the behaviour of many mechanical objects: a wheeled mobile robot, a asynchronous electric motor with high-gain loop currents, and a rigid body with two controllable velocity-correction parameters. The system has been investigated in various publications [4–7]. Problems of stabilizing the Brockett integrator by means of discontinuous control laws have been considered [4], and exponential estimates have been proposed for the approach of the system to equilibrium. In the context of proximal analysis, it has been proved that the Brockett integrator has no continuous stabilization strategies [5]. An interior estimate has been obtained for the attainability set of the system [6].

Control problems for mechanical systems were studied in [7, 8], whose results will be used below to investigate closed optimal trajectories of motion of the object.

The problem will be investigated using methods of optimal control theory [1, 2, 9] and the theory of differential games [10–12]. At the root of the construction are trajectories that steer the system to the boundary of the attainability set [1, 2, 9, 10]. The necessary conditions of the Pontryagin Maximum Principle are derived for these trajectories [2]. In parallel with the construction of attainability sets, the problem of the analytical computation of the value function will be solved. Necessary and sufficient optimality conditions, in the form of Subbotin's differential inequalities [11–15], will be verified for analytical-formulae of the value function.

1. FORMULATION OF PROBLEM 1: CONSTRUCTION OF THE ATTAINABILITY SET AND ANALYSIS OF ITS PROPERTIES

Consider the Brockett integrator, that is, the controlled system

$$\dot{X}_1 = U_1, \quad \dot{X}_2 = U_2, \quad \dot{X}_3 = X_1U_2 - X_2U_1, \quad X(0) = (0, 0, 0) \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 68, No. 5, pp. 707–724, 2004.

0021-8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.09.001

$$|U_1| \le 1, \quad |U_2| \le 1 \tag{1.2}$$

where $X = (X_1, X_2, X_3)$ is the phase vector of the system, $U_j = U_j(t)$ $(j = 1, 2)$ are the controls and $t \in [0, \infty)$ is the time.

It is required to find the attainability set $A = A(T)$ of the system at time T, $0 \le t \le T$, $T \in [0, \infty)$, that is, to determine the set of points $(X_1(T), X_2(T), X_3(T))$ at which system (1.1) will arrive at time T under arbitrarily chosen measurable controls

$$U_j = U_j(t), \quad |U_j(t)| \le 1, \quad 0 \le t \le T, \quad j = 1, 2$$

Definition 1. Measurable functions $U_1(t)$ and $U_2(t)$ that satisfy conditions (1.2) will be called admissible controls.

Definition 2. A trajectory $X(t)$, $t \in [0, T]$, of system (1.1) generated by an admissible control will be called an admissible trajectory.

By an admissible trajectory $(X_1(t), X_2(t))$, $t \in [0, T]$ in the X_1, X_2 plane we mean the projection of an admissible trajectory $X(t)$ onto the X_1, X_2 plane.

According to constraints (1.2), the set of admissible values of the vector $U_1(t)$, $U_2(t)$ is a square with centre at the origin and side equal to 2 (which we denote by P, and by ∂P the boundary of the square P).

Properties of trajectories

Property 1. The projection of the attainability set onto the X_1, X_2 plane is a square K with centre at the origin and side $2T$.

Proof. The coordinate $X_1(T)$ satisfies the relation

$$X_1(T) = \int_0^T U_1(t)dt$$

By the first condition (1.2), $-T \le X_1(T) \le T$, and similarly for $X_2(T)$.

To construct the attainability set, it will suffice to find, for every point (X_1, X_2) in the square K, a set $\{X_1(T), X_2(T), X_3(T): X_1(T) = X_1, X_2(T) = X_2\}$ at which the system may arrive at time T.

Definition 3. The resultant displacement vector (denoted by V_R) is defined as the vector in the X_1, X_2 plane connecting the origin (0, 0) to the final state of the trajectory $(X_1(T), X_2(T))$.

Let $S = (X_1(T), X_2(T))$ be an admissible trajectory generated by an admissible control $(U_1(t), U_2(t))$. Complete S to a closed curve L by a segment ZO connecting the point $(X_1(T), X_2(T))$ to the origin O, as shown in Fig. 1. Let D be the domain bounded be L. The symbol $L^+(L^-)$ will denote the curve L described in the counterclockwise (clockwise) sense.

Property 2. Trajectories $X(T)$, $t \in [0, T]$ satisfy the following integral relations

$$X_3(T) = -2\int_L X_2 dX_1, \quad X_3(T) = -2\iint_D dX_2 dX_1$$

Proof. By Green's formula

$$\iint_D dX_2 dX_1 = -\int_{L^+} X_2 dX_1 = \int_{L^-} X_2 dX_1 = \int_S X_2 dX_1 + \int_{ZO} X_2 dX_1 = \int_S X_2 dX_1 - \frac{X_1(T)X_2(T)}{2} \tag{1.3}$$

Integrating the equations of dynamics (1.1) by parts, we transform the expression for $X_3(T)$ as follows:

$$X_3(T) = X_1(T)X_2(T) - 2\int_0^T X_2(t)dX_1(t) = 2\iint_D dX_2 dX_1 = -2\int_L X_2 dX_1 \tag{1.4}$$

where we have used relations (1.3).

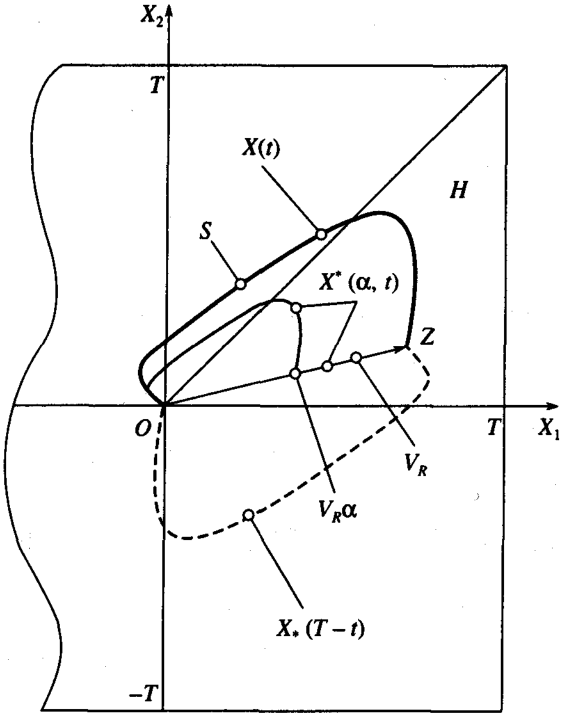

Fig. 1

The quantity $X_3(T)$ is defined, apart from the sign, by the area of the figure bounded by the curve L.

If V_R is the zero vector, the curves L and S in the last integral of (1.4) coincide.

We will consider the following symmetric transformations of the admissible trajectory S: symmetries S_1 and S_2 relative to the coordinate angles axes X_1 and X_2 symmetries S_3 and S_4 relative to the bisectors of the first and second coordinate angles, and central symmetry S_5 relative to the midpoint of the resultant displacement vector.

Proposition 1. The symmetries $S_1, \ldots, S_5$ are admissible trajectories in the X_1, X_2 plane.

Proof. Let $(U_1(t), U_2(t))$ be an admissible control. To construct S_3, we need only take the symmetric control

$$U_1^*(t) = U_2(t), \quad U_2^*(t) = U_1(t) \tag{1.5}$$

as the generator of an admissible control. This may be done because U_1 and U_2 have the same sets of admissible values. To construct S_4, we just switch the signs of both coordinates of Eq. (1.5). To construct $S_1(S_2)$, reverse the sign of the first (second) coordinate of control (1.5).

To construct S_5, we choose the control

$$U_1^*(t) = U_1(T-t), \quad U_2^*(t) = U_2(T-t)$$

The points $X_*(T-t)$ and $X(t)$ are symmetric with respect to this mapping (Fig. 1).

We will show that the admissible path S_5 is indeed symmetric to the original path S relative to the midpoint of the vector V_R. Such a symmetry must satisfy the relation

$$X_* = V_R/2 + (V_R/2 - X) = V_R - X$$

The truth of this relation follows from the definition of $U^*(t)$. Indeed,

$$X^*(T-t) = \int_0^{T-t} U^*(\tau)d\tau = \int_0^{T-t} U(T-\tau)d\tau =$$

$$= \int_0^T U(T-\tau)d\tau - \int_t^T U(T-\tau)d\tau = X(T) - X(t)$$

Corollary. Proposition 1 implies that the third coordinate of the motion for the symmetry S_5 satisfies the symmetry formula $X_3^*(T) = -X_3(T)$. Due to this symmetry, we need only construct the attainability set for positive (or negative) third coordinates.

Proof. Figures bounded by symmetric trajectories have equal areas. By Property 2, the absolute value of $X_3(T)$ equals twice the area of the figure bounded by the completed trajectory of motion. If the direction of motion is changed, the sign of $X_3(T)$ is reversed.

Let H be the set of points of the square K below or on the bisector of the first coordinate angle in the first quadrant (Fig. 1). Fix a point $Z \in H$. Consider the trajectories $(X_1(t), X_2(t), X_3(t))$ of the system such that the endpoint of the resultant sector V_R in the X_1, X_2 plane is Z.

Let $M = M(Z)$ be the maximum of the set of the third coordinates $X_3(T)$ of the points $(X_1(T), X_2(T), X_3(T))$ of such trajectories.

Property 3. All the points of the segment $I(Z)$ whose endpoints are $(Z, M(Z))$ and $(Z, 0)$ lie in the attainability set $A(T)$.

Proof. Let $U(t)$ be a control that steers the system from the origin (0, 0, 0) to the point $(Z, M(Z))$ and let $X(t) = (X_1(t), X_2(t))$ be the projection of the corresponding trajectory onto the X_1, X_2 plane. Construct a control $U^* = U^*(\alpha, t)$ that steers the system to any point of the segment $I(Z)$. For such a control we have $V_R = Z$, $X_3(T) = \alpha M(Z)$, $0 \le \alpha \le 1$. We put $\tilde{t} = t/\alpha$, $\tilde{T} = T/\alpha$. Suppose

$$U^*(\alpha, t) = \begin{cases} U(\tilde{t}), & 0 \le \tilde{t} \le T \\ V_R/(T - \alpha T), & T < \tilde{t} \le \tilde{T} \end{cases}$$

Then the projection of the corresponding trajectory onto the X_1, X_2 plane is defined by

$$X^*(\alpha, t) = \begin{cases} \alpha X(\tilde{t}), & 0 \le \tilde{t} \le T \\ V_R t/T, & T < \tilde{t} \le \tilde{T} \end{cases}$$

Accordingly, the area of the figure D^* bounded by the trajectory $X^*(\alpha, t)$ and the resultant displacement vector equals the area of the figure D multiplied by α (see Fig. 1). By Property 2

$$X_3^*(\alpha, T) = -2\iint_{D^*} dX_2 dX_1 = \alpha\left(-2\iint_D dX_2 dX_1\right) = \alpha X_3(T)$$

Let us assume that the quantities $M(Z)$ have been constructed for all points Z of H. By Property 3 and the corollary to Proposition 1, the set of points

$$A(H) = \{(Z, m)\colon Z \in H, -M(Z) \le m \le M(Z)\}$$

is contained in the attainability set A.

Property 4. The attainability set A may be constructed on the basis of symmetry properties from the set $A(H)$, by means of the following algorithm: First, applying the symmetry S_1 to $A(H)$, construct the attainability set in the first quadrant; then, applying the symmetry S_2, in the second quadrant; and finally, applying the symmetry S_3, in the third and fourth quadrants.

Thus, the problem of constructing attainability sets reduces to determining a control that will steer the system from the origin to the point $(Z, M(Z))$ or, symmetrically, the point $(Z, -M(Z))$. In other words, it required to find a control $U(t)$ that will maximize (minimize) $X_3(T)$ as a functional of the corresponding trajectories $(X_1(t), X_2(t), X_3(t))$ of the system. By Property 2, minimization of the functional $X_3(T)$ is equivalent to the following control problem (Problem 2)

$$I = \int_0^T X_2 dX_1 \to \max \tag{1.6}$$

$$\begin{aligned} &\dot{X}_1 = U_1, \quad \dot{X}_2 = U_2; \quad |U_1| \le 1, \quad |U_2| \le 1 \\ &X_1(0) = X_2(0) = 0, \quad (X_1(T), X_2(T)) \in H \end{aligned} \tag{1.7}$$

In this two-dimensional control problem, the initial point of the trajectory is fixed at the origin O and the right end at an arbitrary point $(X_1(T), X_2(T))$ of the set H.

We recall that, according to optimal control theory [2], any part of an optimal trajectory is also optimal. On the basis of this property, the solution of Problem 2 can be reduced to analysing the special case of Problem 2 with a closed trajectory. Let us call this special case Problem 3. The corresponding system (which we call System A) differs from system (1.7) in that the condition $(X_1(T), X_2(T)) \in H$ is replaced by $X_1(T) = X_2(T) = 0$.

Problem 3 is to find a control that will steer System A from the origin to the origin and maximize the functional I of (1.6).

2. THE SOLUTION OF PROBLEM 3

Let $U(t) = (U_1(t), U_2(t))$ be an admissible control defined in the interval $[0, T]$, and let $X(t) = (X_1(t), X_2(t))$ be the corresponding trajectory of motion of System A. Let $U^*(s) = (U_1^*(s), U_2^*(s))$, $s \in [0, S]$ denote a control lying on the boundary of the square P and generating a trajectory $X^*(s) = (X_1^*(s), X_2^*(s))$ so that the images of the trajectories $X(t)$ and $X^*(s)$ coincide, that is,

$$\{X(t), 0 \le t \le T\} = \{X^*(t), 0 \le t \le T\}$$

Proposition 2. A control $U^*(s)$, $s \in [0, S]$ may be constructed for any control $U(t)$, $t \in [0, T]$. Under these conditions $S \le T$.

Proof. Let $t \in [0, T]$. Fix $x \in [0, T]$. Choose a function $\lambda(x) \ge 1$ such that $\lambda(x)U(x) \in \partial P$, and define a new variable

$$y = y(x) = \int_0^x \frac{d\xi}{\lambda(\xi)}$$

The new time variable is defined by $s = y(t)$. Moreover, $y = y(x)$ is a strictly increasing function, so that its inverse $x = x(y)$ exists. Make the change of variables $x = x(y)$ in the definite integral for the trajectory of motion of System A. We have

$$X(t) = \int_0^t U(x)dx = \int_0^t U(x)\lambda(x)\frac{dx}{\lambda(x)} = \int_0^s U(x(y))\lambda(x(y))dy = \int_0^s U^*(y)dy = X^*(s)$$

Thus, the required control U^* on the boundary of the square P is defined by $U^*(y) = U(x(y))\lambda(x(y))$.

Proposition 3. Problem 3 is equivalent to a problem in which the control $U(t) = (U_1(t), U_2(t))$ lie on the boundary of the square P, that is

$$\max\{|U_1(t)|, |U_2(t)|\} = 1$$

Proof. Let I_1 be the optimal value of the function I in Problem 3 with $U(t) \in P$, and let I_2 be the optimal value of the function I for the analogous problem with $U(t) \in \partial P$. Clearly, $I_2 \le I_1$, since $\partial P \subset P$.

We will show that $I_2 \geq I_1$. Let $U(t)$ be an optimal control for which the value I_1 of the functional I in Problem 3 is a maximum. It follows from Proposition 2 that the trajectory generated by a control $U(t) \in P, t \in [0, T]$, may be generated by a control $U^*(s) \in \partial P, s \in [0, S]$, with $S \leq T$. We define a control $U^*(s)$ in the interval $(S, T]$ as follows:

$$U^*(s) = \begin{cases} U^*(S), & s \in (S, (S+T)/2] \\ -U^*(S), & s \in ((S+T)/2, T] \end{cases}$$

The $X^*(T) = X^*(S)$ and $\int_S^T X_2^* dX_1^* = 0$. Consequently, by construction of $U^*(s), s \in [0, T]$, we have the chain of relations

$$I_1 = \int_0^T X_2 dX_1 = \int_0^S X_2^* dX_1^* = \int_0^T X_2^* dX_1^* \leq I_2$$

We finally obtain $I_1 = I_2$.

Proposition 4. Problem 3 is equivalent to a problem in which the controls $U(t) = (U_1(t), U_2(t))$ lie for almost all t at the vertices of the square P:

$$|U_1(t)| = 1, \quad |U_2(t)| = 1$$

Proof. By Proposition 3, we may consider an equivalent problem in which $U(t) \in \partial P$. Let I_2 be the optimal value of the functional I in that problem and let I_3 be the optimal value of the functional I for a problem in which $U(t)$ lies at the vertices of the square. Clearly, $I_3 \leq I_2$.

We will show that $I_3 \geq I_2$. Let $U(t)$ be an optimal control which maximizes the value of I_2. Let T_1 denote the set of points $t \in [0, T]$ for which $U_1(t) = 1$. By definition of a measurable function, the set T_1 is measurable. We can assume without loss of generality that the measure $\mu(T_1)$ of the set T_1 is positive. Let $n \geq 1$ be a given number, and consider a partition of the interval $[0, T]$ with diameter $(0.1)^n T$. Let J_n denote the set of subintervals of the partition that lie entirely in T_1. Let d_n be the sum of lengths of the subintervals in J_n. By the definition of measure, $d(J_n) \uparrow \mu(T_1)$ as $n \to \infty$. Beginning with some number n, the set J_n is not empty. Indeed, fix $\varepsilon = \mu(T_1)/2$; then, by the definition of limit, $n(\varepsilon)$ exists such that, for all $n \geq n(\varepsilon)$, we have the inequality $\mu(T_1) - d(J_n) < \mu(T_1)/2$, that is, $d(J_n) > \mu(T_1)/2 > 0$.

Consider an interval $J = [t_1, t_2]$ in the set J_n. We have $U_1(t) = 1$, $|U_2(t)| \leq 1, t \in J$. The motion of the system occurs toward the right from the initial point $X(t_1) = (X_1(t_1), X_2(t_1))$ to the final point $X(t_2) = (X_1(t_2), X_2(t_2))$. Construct a trajectory $X^*(t) = (X_1^*(t), X_2^*(t))$ over the interval $[t_1, t_2]$ which has the same initial and final points but is generated by the control

$$U_1(t) = 1, \quad U_2(t) = \begin{cases} 1, & t \in [t_1, \tau) \\ -1, & t \in [\tau, t_2) \end{cases}$$

where the switching time τ is defined by

$$t_1 + (\Delta T + \Delta X_2)/2 = (t_1 + t_2 + X_2(t_2) - X_2(t_1))/2$$

The trajectories $X(t), X^*(t)$ are shown in Fig. 2. For these trajectories,

$$dX_1 = dX_1^* = U_1(t)dt = dt > 0, \quad X_2^*(t) \geq X_2(t), \quad t \in [t_1, t_2]$$

These relations imply the inequality

$$\int_{t_1}^{t_2} X_2^*(t) dX_1^*(t) \leq \int_{t_1}^{t_2} X_2(t) dX_1(t)$$

Taking the limit as $n \to \infty$ in the partition of the interval $[0, T]$, we obtain

$$\int_{T_1} X_2^*(t) dX_1^*(t) \leq \int_{T_1} X_2(t) dX_1(t)$$

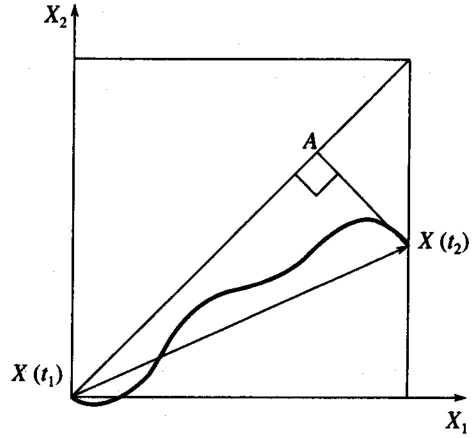

Fig. 2

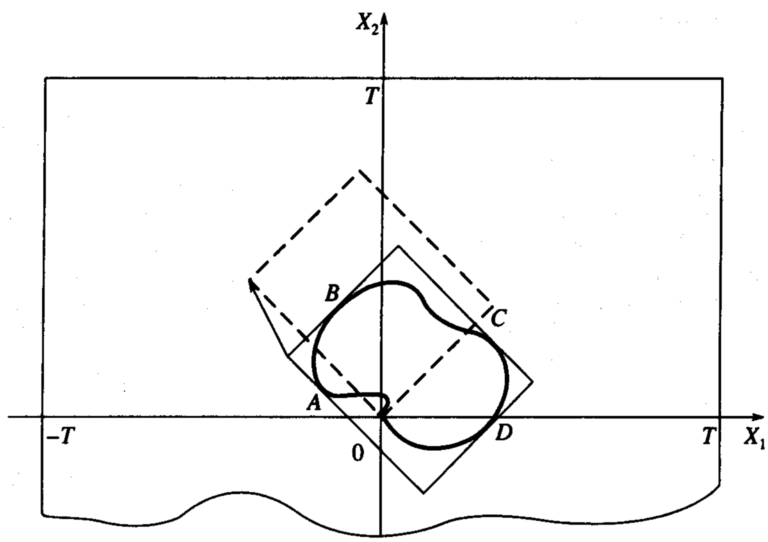

Fig. 3

Analogous arguments can be used for the set

$$T_2 = \{t: t \in [0, T], U_2(t) = -1\} = [0, T]\backslash T_1$$

Combining these arguments, we obtain $I_2 \leq I_3$, and hence $I_2 = I_3$.

Let Φ denote the figure bounded by a trajectory $X(t)$ in Problem 3 that has the maximum area among all such figures.

Proposition 5. The optimal trajectory of Problem 3 is the boundary of a rectangle with given perimeter $T\sqrt{2}$.

Proof. We will show that, for any figure Φ, generated by an admissible control $U(t)$, there is a rectangle Φ^*, generated by admissible control $U^*(t)$ and with area at least that of Φ. The curve in Fig. 3 represents the boundary of Φ. The rectangle Φ^* is constructed, for example, in two steps: first, draw a circumscribed rectangle of least area around Φ with sides parallel to the bisectors of the coordinate angles; then displace the rectangle parallel to itself until its lower corner is at the origin.

Let A, B, C, D denote arbitrary points at which the different sides of the rectangle touch the figure Φ (Fig. 3). Consider the point B and C. As in the proof of Proposition 4, we construct an admissible control $U^*(t)$ that steers the point B to C along the boundary of the rectangle in a time not exceeding the time of motion along the original trajectory. In the process, the area of the figure described by the new trajectory can only increase. Similar reasoning holds for other combinations of adjacent points. We thus conclude that the area of the rectangle Φ* is no less than that the figure Φ.

Of all rectangles having the same perimeter, the square has the greatest area. The solution of Problem 3 is given by squares whose sides are parallel to the bisectors of the coordinate angles. The origin may lie at any point on the boundary of the square.

Thus, Problem 3 is similar to Dido's problem [16]. We recall that Dido's problem [also known as the isoperimetric problem] is the classical problem of the calculus of of variations: to find a curve of given length bounding the figure of maximum area. The solution is given by a two-parameter family of circles; one parameter defines the length of the circle, the other, the position of the centre of the circle in the plane.

By analogy with the solution of Dido's problem, the solution Problem 3 may also be described by a two-parameter family – not of circles but of the perimeters of squares. The parametric representation of the optimal trajectories has the form

$$X_1(T, a_0, a(t)) = \frac{T}{2}[\cos^2(a_0 - a)\operatorname{sign}(\cos(a - a_0)) - \cos^2 a_0 \operatorname{sign}(\cos(-a_0))]$$

$$X_2(T, a_0, a(t)) = \frac{T}{2}[\sin^2(a_0 - a)\operatorname{sign}(\sin(a - a_0)) - \sin^2 a_0 \operatorname{sign}(\sin(-a_0))]$$

$$a = a(t) \in [0, 2\pi], \quad a_0 \in [0, 2\pi]$$

where T and a_0 are the parameters of the family. The parameter T defines the length of the trajectory, and the parameter a_0, the position of the centre of the square in the plane. The parameter t is the time of motion along the trajectory. The function $a(t)$ is the angle between the radius-vector connecting the centre of the square to the origin and the radius-vector connecting the centre of the square to the point $X(T)$ of the trajectory (Fig. 4). This parameter must satisfy the following differential equalities at points of differentiability

$$|d\cos^2(a_0 - a(t))/dt| = 1, \quad |d\sin^2(a_0 - a(t))/dt| = 1$$

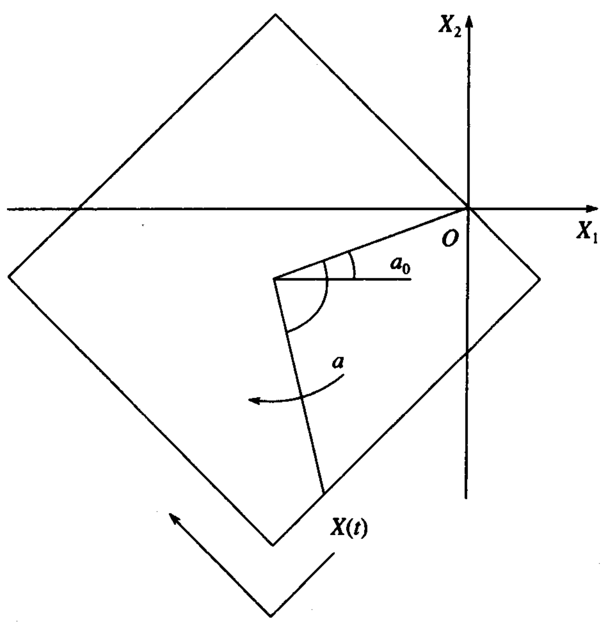

Fig. 4

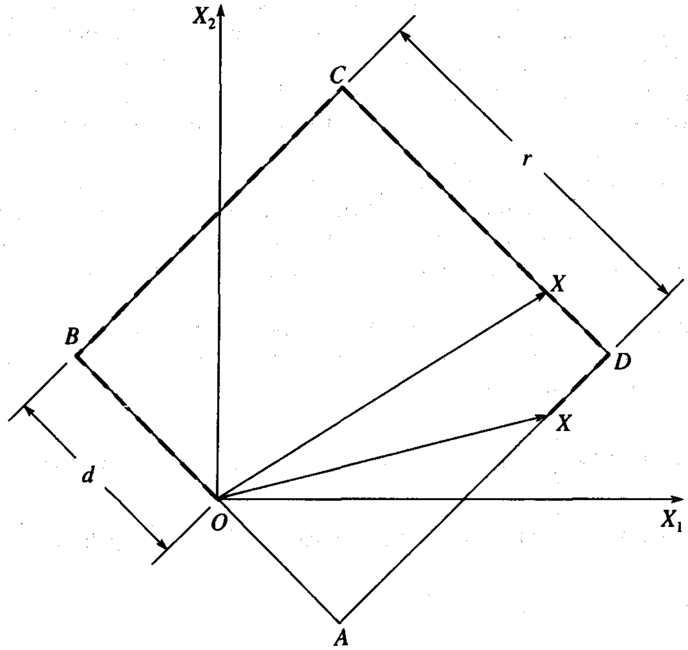

Fig. 5

3. THE SOLUTION OF PROBLEM 2

We will look for trajectories solving Problem 2 as parts of the trajectories that solve problem 3. Fix a point $X \in H$. Let T be the time of motion from the origin to X in Problem 2, i.e. $X(T) = X$. We must select parameters $T^* \geq T$, a_0 for which the trajectory of the solution of Problem 3 passes through X and the time of motion from the origin to X equals T.

Using Property 2, the integral I equals the area bounded by the resultant displacement vector and the trajectory. Consider a square bounded by an optimal trajectory of Problem 3. Its sides are parallel to the bisectors of the coordinate angles, and its vertices, in clockwise order, are denoted by A, B, C, D, beginning with the lowest vertex. We recall that the length of the side of this square is $r = \sqrt{2}T^*/4$. Let us consider all possible trajectories solving Problem 3 that run along the perimeter of a square with side r from the origin O to the point X in time T, assuming that the origin lies on the side AB of the square $ABCD$.

In this case, all possible positions of such squares are described by two parameters: the length r of the side of the square $ABCD$ and the distance d from the point O to the vertex A.

In addition, there are two possible positions of the resultant displacement vector for these trajectories. In the first version, the trajectory runs along three sides of the square, being made up of the segments OB, BC and CX. In the second, the trajectory runs along all four sides of the square and is made up of the segments OB, BC, CD and DX (Fig. 5).

Consider the right-angled triangle OXW whose hypotenuse is the vector OX, and whose legs OW and XW lie below the hypotenuse and are parallel to the bisectors of the first and second coordinate angles.

Proposition 6. The length of the legs OW and XW of the triangle OXW are determined by the coordinates of the vector X and given by

$$|OW| = \zeta_-, \quad |XW| = \zeta_+; \quad \zeta_\pm = (X_1 \pm X_2)/\sqrt{2}$$

The proof is obvious.

The parameters r and d must be determined from two equations defining the final point X of the trajectory and the time T of motion along it. Using Proposition 6, we determine the parameters r and d for both possible positions of the resultant displacement vector.

In the first version.

$$r = \zeta_+, \quad \sqrt{2}T = r + 2d + \zeta_-$$

whence $d = (T - X_1)/\sqrt{2}$. In addition, the parameter d must satisfy the inequality $d \le r - \zeta$. It follows from these relations that a solution of this type exists only for points of the triangle H that satisfy the inequality $X_2 \ge (T - X_1)/2$.

In the first version, the area of the figure bounded by the resultant displacement vector and trajectory of motion equals the area of the quadrilateral $OBCX$ and is computed by the formula

$$I_1 = rd + r\zeta_-/2 = (\sqrt{2}T - \zeta_+)\zeta_+/2 \tag{3.1}$$

Similarly, for the second version, the parameter r and d must be determined by two equations defining the final point X and the time T of motion along it. Then

$$\sqrt{2}T = 4r - (\zeta_- + \zeta_+), \quad d = r - \zeta_- \Rightarrow r = (T + X_1)/(2\sqrt{2}), \quad d = T/(2\sqrt{2}) + \zeta_+$$

In addition, the parameter r must satisfy the inequality $r \ge \zeta_+$. It follows from these relations that a solution of this type exists only for points of the triangle H that satisfy the inequality $X_2 \le (T - X_1)/2$.

In the second version, the area of the figure bounded by the resultant displacement vector and the trajectory of motion equals the area of the pentagon $OBCDX$ and is computed by the formula

$$I_2 = r^2 - \zeta_-\zeta_+/2 = (T^2 + 2TX_1 - X_1^2 + 2X_2^2)/8 \tag{3.2}$$

The solution of Problem 2 depends on the coordinates $(X_1(T), X_2(T)) = (X_1, X_2) \in H$ of the right end of the trajectory of the system and may be expressed as

$$I(T, X_1, X_2) = \begin{cases} I_1(T, X_1, X_2), & X_2 > (T - X_1)/2 \\ I_2(T, X_1, X_2), & X_2 \le (T - X_1)/2 \end{cases} \tag{3.3}$$

Consequently, by Property 2, the upper boundary of the attainability set in the triangle H is the graph of the function

$$X_3(T, X_1, X_2) = 2I(T, X_1, X_2) \tag{3.4}$$

The function $I(T, X_1, X_2)$ is determined by formulae (3.1)–(3.3).

Using Property 5, the boundaries of the attainability sets for $T = 4$ have been constructed for $T = 4$ (Fig. 6).

4. VERIFICATION OF THE NECESSARY CONDITIONS FOR OPTIMALITY OF THE TRAJECTORIES

We will now verify that the trajectories obtained by solving Problem 3 satisfy Pontryagin's Maximum Principle [2] for Problem 1. We rewrite problem 1 for the lower boundary of the attainability domain as

$$\begin{gathered} I = \int_0^T (X_1U_2 - X_2U_1)dt \to \min \\ \dot{X}_1 = U_1, \quad \dot{X}_2 = U_2; \quad |U_1| \le 1, \quad |U_2| \le 1; \quad X_1(0) = X_1(T) = X_2(0) = X_2(T) = 0 \end{gathered} \tag{4.1}$$

The system of equations of the Maximum Principle for the auxiliary conjugate variables ψ_i is

$$\dot{\psi}_i = -\sum_{\alpha = 0}^{2} \frac{\partial f^\alpha(X, U)}{\partial X_i}\psi_\alpha$$

Taking into account that

$$f^1(X, U) = U_1, \quad f^2(X, U) = U_2, \quad f^0(X, U) = X_1U_2 - X_2U_1$$

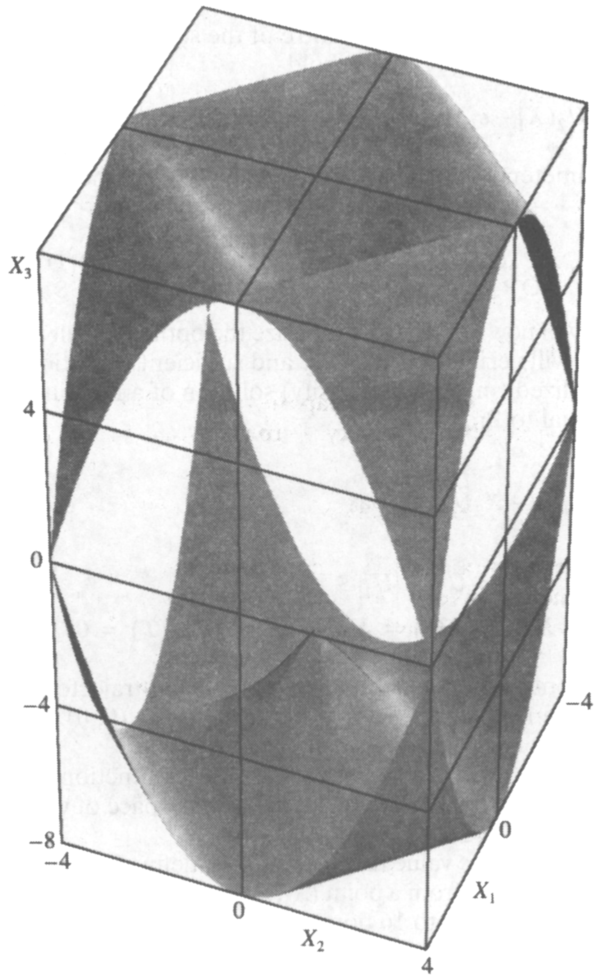

Fig. 6

we have

$$\dot{\psi}_1 = -U_2\psi_0, \quad \dot{\psi}_2 = U_1\psi_0, \quad \dot{\psi}_0 = 0 \tag{4.2}$$

Integrating this system, we get

$$\psi_1 = -(X_2 + c_1)c, \quad \psi_2 = (X_1 + c_2)c, \quad \psi_0 = c \tag{4.3}$$

By the conditions of the Maximum Principle for a problem with fixed time, we must choose the negative value for the constant x.

The Hamiltonian of system (4.1) is

$$\begin{aligned} H(\psi, X, U) &= \sum_{\alpha=0}^{2} f^{\alpha}(X, U)\psi_{\alpha} = \\ &= (X_1U_2 - X_2U_1)c - U_1(X_2 + c_1)c + U_2(X_1 + c_2)c \end{aligned} \tag{4.4}$$

A necessary condition for the Maximum Principle to be valid is that

$$H(\psi, X, U) = \max_{V \in P} H(\psi, X, V), \quad U = (U_1, U_2), \quad V = (V_1, V_2)$$

and this condition in turn transforms into the equality

$$U_2(2X_1 + c_2) - U_1(2X_2 + c_1) = |2X_2 + c_1| + |2X_1 + c_2| \tag{4.5}$$

We will verify that the optimal trajectories of Problem 3 satisfy this relation. Indeed, the optimal trajectory is the boundary of a square; let the centre of the square have coordinates (c_{k1}, c_{k2}). Clearly, for optimal trajectories the following relations hold

$$U_2(X_1 - c_{k1}) - U_1(X_2 - c_{k2}) = |X_1 - c_{k1}| + |X_2 - c_{k2}|$$

Now, choosing the parameters $c_1 = -2c_{k1}$, $c_2 = -2c_{k2}$, we obtain the required relation (4.5).

5. VERIFICATION OF THE SUFFICIENT CONDITIONS FOR OPTIMALITY OF TRAJECTORIES

Our solution of Problem 2 will now be used to synthesize the optimal result function (the value function). For the value function, we will verify the necessary and sufficient conditions – differential inequalities [11] satisfied by the generalized (minimax, viscosity) solution of a Hamilton–Jacobi equation.

Consider the problem dual to Problem 2:

$$\begin{aligned} &I = \int_{T_0}^{T} (X_1U_2 - X_2U_1) \to \max \\ &\dot{X}_1 = U_1, \quad \dot{X}_2 = U_2; \quad |U_1| \le 1, \quad |U_2| \le 1 \\ &X_1(T_0) = X_1, \quad X_2(T_0) = X_2, \quad X_1(T) = X_2(T) = 0; \quad 0 \le T_0 \le T \end{aligned} \tag{5.1}$$

In this two-dimensional control problem, the initial point of the trajectory is fixed at an arbitrary point in the (X_1, X_2) plane, and the right endpoint is fixed at the origin (0, 0). The solution of this problem is obtained from the solution of Problem 2.

We set up the value function for the problem (5.1) from the function (3.4). It takes finite values in the controllability domain of system (5.1), which is cone in the space of variables X_1, X_2 and T, defined by the relation $|X_1| \le T - T_0$, $|X_2| \le T - T_0$.

Outside the controllability cone, the value function must be defined as minus infinity, since the problem of steering a trajectory of system (5.1) from a point (X_1, X_2) to the origin (0, 0) in time $T - T_0$ is unsolvable.

The controllability cone is divided into 16 domains, in each of which an analytical formula has been found for the value function. We will present expressions for the value function $W(X_1, X_2, T - T_0)$ at the points of the domain $\{(X_1, X_2, T)\colon 0 \le X_1 \le T - T_0, 0 \le X_2 \le T - T_0, X_2 \le X_1\}$ corresponding to the triangle H. We introduce the following notation

$$\xi_j = (T - T_0 - X_j)/2, \quad \varphi = \partial\Phi/\partial X, \quad \varphi_0 = \partial\Phi/\partial T_0, \quad \varphi_j = \partial\Phi/\partial X_j; \quad j = 1, 2$$

We have

$$\Phi(X_1, X_2, T - T_0) = \begin{cases} (\xi_1 + \xi_2)(X_1 + X_2), & \text{if} \quad X_2 > \xi_1 \\ (\xi_1 + X_1)^2 - (X_1^2 - X_2^2)/2, & \text{if} \quad X_2 \le \xi_1 \end{cases}$$

For the other domains, the value function W can then be determined using the symmetries of Proposition 1 (see Fig. 7).

For the corresponding domains and the value function W, the Hamilton–Jacobi equation

$$H(X, \varphi) + \varphi_0 = 0$$

must hold, where

$$H(x, \varphi) = \max_{U_1, U_2} (U_1\varphi_1 + U_2\varphi_2 + X_1U_2 - X_2U_1) = |\varphi_1 - X_2| + |\varphi_2 + X_1|$$

For domain 1 (Fig. 7) we have

$$\varphi_1 = \xi_1, \quad \varphi_2 = X_2, \quad \varphi_0 = -\xi_1 - X_1$$

and the Hamilton–Jacobi equation is obviously satisfied. The validity of the Hamilton–Jacobi equation for domain 2 can be verified similarly.

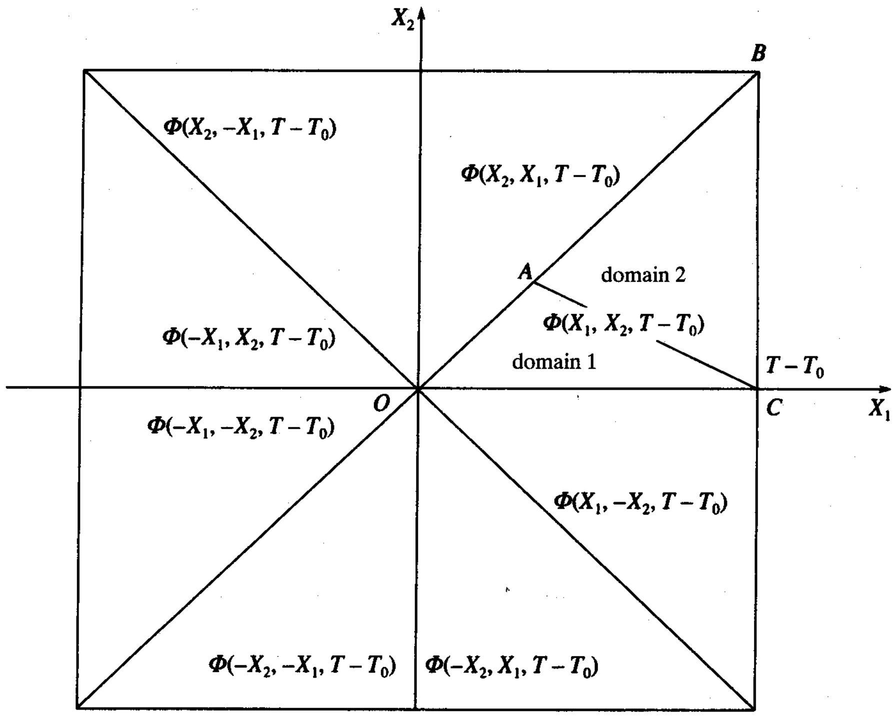

Fig. 7

The 16 domains under consideration, glued together two by two, yield 20 surfaces. The situation is pictured schematically in Fig. 7 in two dimensions, with the glued surfaces represented by segments. For the 12 surfaces OC, AC, AB and their symmetric images, the gluing operation is smooth. For the four surfaces OA and its symmetric images, the gluing is not smooth, but involves the operation of taking the maximum of two functions. For example, on the surface OA, the maximum operation is used to glue together the two functions $\Phi(X_1, X_2, T - T_0)$ and $\Phi(X_2, X_1, T - T_0)$.

On four lines, namely, A and its symmetric images, four functions are smoothly glued together.

On the lines associated with the origin O, 8 functions are glued together. Of these, four are glued together smoothly, forming for smooth functions, which are then glued together non-smoothly using the maximum operation.

On the boundary of the controllability cone, functions taking a finite value are glued together with functions taking the value minus infinity. The gluing operation takes places in these cases on the faces and edges of the cone.

Let us verify that the functions on the surfaces OC, AC and AB are glued together smoothly.

On the surface OC ($X_2 = 0$), the gradients of the functions being glued together, $F_1 = \Phi(X_1, X_2, T - T_0)$ and $F_2 = \Phi(X_1, -X_2, T - T_0)$, satisfy the relations

$$\frac{\partial F_1}{\partial X_1} = \frac{\partial F_2}{\partial X_1} = \xi_1, \quad \frac{\partial F_1}{\partial X_2} = X_2\Big|_{OC} = 0 = -X_2\Big|_{OC} = \frac{\partial F_2}{\partial X_2}, \quad \frac{\partial F_1}{\partial T_0} = \frac{\partial F_2}{\partial T_0} = -\xi_1 - X_1$$

On the surface AC ($X_2 = \xi_1$), the components of the function Φ are glued together smoothly:

$$F_1 = (\xi_1 + \xi_2)(X_1 + X_2), \quad F_2 = (\xi_1 + X_1)^2 - (X_1^2 - X_2^2)/2 \tag{5.2}$$

Now let us consider the surface AB, on which the functions constructed using the symmetry S_3 are glued together. On this surface the function Φ is equal to $(\xi_1 + \xi_2)(X_1 + X_2)$ and itself satisfies the symmetry property S_3. In the neighbourhood of the surface AB, therefore, the value function is described by the values of a smooth function Φ which, as shown previously, satisfies a Hamilton–Jacobi equation.

We will now verify that on the surface OA, where the two functions

$$F_1 = (\xi_1 + X_1)^2 - (X_1^2 - X_2^2)/2, \quad F_2 = (\xi_2 + X_2)^2 - (X_2^2 - X_1^2)/2$$

are glued together non-smoothly using the maximum operation, the generalized gradients [11] of the value function satisfy inequalities that replace the Hamilton–Jacobi equations:

$$H(x, \varphi) + \varphi_0 \le 0, \quad \text{where} \quad (\varphi, \varphi_0) \in D^-\Phi(T, T_0, X) \tag{5.3}$$

$$H(x, \varphi) + \varphi_0 \ge 0, \quad \text{where} \quad (\varphi, \varphi_0) \in D^+\Phi(T, T_0, X) \tag{5.4}$$

The sets D^+ and D^- are the superdifferential and subdifferential, respectively, of the function F, defined by

$$D^+\Phi(T, T_0, X) = \{p | \partial\Phi_e(T, T_0, X) \ge \langle p, e\rangle \quad \text{for all} \quad e\}$$

$$D^-\Phi(T, T_0, X) = \{p | \partial\Phi_e(T, T_0, X) \le \langle p, e\rangle \quad \text{for all} \quad e\}$$

where $\partial\Phi_e(T, T_0, X)$ is the derivative of the function Φ in the direction $e = (e_{T_0}, e_X)$

$$\partial\Phi_e(T, T_0, X) = \lim_{\delta \downarrow 0} \frac{\Phi(T, T_0 + \delta e_{T_0}, X + \delta e_X) - \Phi(T, T_0, X)}{\delta}$$

On the surface OA

$$\partial\Phi_e(T, T_0, X) = \max_{j = 1, 2} \langle \nabla F_j, e\rangle = (-\xi_1 - X_1)e_{T_0} + \max\{\xi_1 e_1 + X_2 e_2, X_1 e_1 + \xi_1 e_2\}$$

$$\nabla F_i = \left(\frac{\partial F_j}{\partial T_0}, \frac{\partial T_j}{\partial X_1}, \frac{\partial F_j}{\partial X_2}\right), \quad F_j = F_j(T, T_0, X)$$

The subdifferential is given by

$$D^-\Phi(T, T_0, X) = \text{co}\{\nabla F_1, \nabla F_2\} = \{a\nabla F_1 + (1-a)\nabla F_2 : a \in [0, 1]\}|_{X_1 = X_2 = Y} =$$

$$= \{(-\eta_+, aY + (1-a)\eta_-, a\eta_- + (1-a)Y) : a \in [0, 1], Y \in [0, (T - T_0)/3]\}$$

$$\eta_\pm = (T - T_0 \pm Y)/2$$

The superdifferential of the function Φ on the surface OA is the empty set, and so inequality (5.4) holds automatically.

To verify inequality (5.3), we compute its left-hand side for vectors of the subdifferential $D^-\Phi(T, T_0, X)$:

$$H(x, \varphi) + \varphi_0 = -\eta_+ + |a(T - T_0 - 3Y)/2| + |\eta_+ - a(T - T_0 - 3Y)/2|$$

This expression clearly takes non-positive values, since it vanishes identically at $a \in [0, 1]$, $Y \in [0, [T - T_0)/3]$.

Thus, the differential inequalities (5.3) and (5.4) hold at all points of the surface OA. On the line $O(X_1 = X_2 = 0)$, 4 smooth functions are glued together using the maximum operation. The superdifferential of the resulting function is given by

$$D^-\Phi(T, T_0, X)|_O = \text{co}\{\nabla F_1, \nabla F_2, \nabla F_3, \nabla F_4\}|_O$$

$$\nabla F_{1,3} = (-\xi_1 - X_1, \pm\xi_1, \pm X_2) = (-\tau, \pm\tau, 0),$$

$$\nabla F_{2,4} = (-\xi_1 - X_1, \pm X_2, \pm\xi_1) = (-\tau, 0, \pm\tau); \quad \tau = (T - T_0)/2$$

The subdifferential is a square in the X_1, X_2 plane with centre at the origin and sides parallel to the bisectors of the coordinate angles:

$$D^-\Phi(T, T_0, X)|_{X_1 = X_2 = 0} = \{(\varphi_0, \varphi_1, \varphi_2) : \varphi_0 = -\tau, |\varphi_1| + |\varphi_2| - \tau \le 0\} \tag{5.5}$$

To verify inequality (5.3), we evaluate its left-hand side for vectors of the subdifferential $(\varphi_0, \varphi_1, \varphi_2) \in D^-\Phi(T, T_0, X)$. We have

$$H(x, \varphi) + \varphi_0 = |\varphi_1| + |\varphi_2| - \tau$$

By (5.5), the Hamiltonian takes non-positive values.

The superdifferential of Φ on the line O is the empty set, so that inequality (5.4) holds automatically.

We will now verify that the differential inequalities for the generalized gradients [11] of the value function, replacing the Hamiltonian–Jacobi equation, are also satisfied on the boundary of the attainability set – the surface BC, on which the finite and infinite values of the value function are glued together.

The directional derivative of the value function on the surface BC is

$$\partial\Phi_e(T, T_0, X) = \begin{cases} -\infty, & e_{T_0} + e_{X_1} > 0 \\ \langle \nabla F_1, e \rangle, & e_{T_0} + e_{X_1} \le 0 \end{cases}$$

(the function F_1 is defined by the first formula of (5.2)).

We will now compute the superdifferential of the function Φ. By definition,

$$D^+\Phi(T, T_0, X) = \begin{cases} p\colon -\infty \le \langle p, e \rangle & \text{if} \quad e_{T_0} + e_{X_1} > 0 \\ p\colon \langle \nabla F_1, e \rangle \le \langle p, e \rangle & \text{if} \quad e_{T_0} + e_{X_1} \le 0 \end{cases}$$

Since the upper inequality of the system holds for all vectors P, the superdifferential is defined by the lower inequality only. This inequality must hold at all points in the half-space defined by $e_{T_0} + e_{X_1} \le 0$. Hence, the vectors $\nabla F_1 - p$ are perpendicular to the plane $e_{T_0} + e_{X_1} = 0$ and have the same direction as the vector $(1, 1, 0)$. Then

$$\nabla F_1 - p = (a, a, 0), \quad a \in [0, \infty]$$

Finally, we obtain the following formula for the superdifferential

$$\begin{aligned} D^+\Phi(T, T_0, X)|_{BC} &= \{p\colon p = \nabla F_1 + (-a, -a, 0), a \in [0, \infty]\} = \\ &= \{p\colon p = (-X_1 - X_2 - a, \xi_1 - X_2/2 - a, \xi_1 - X_2/2), a \in [0, \infty]\} \end{aligned} \tag{5.6}$$

The superdifferential is a ray emanating from the point ∇F_1, codirectional with the vector $(-1, -1, 0)$. Then inequality (5.4) may be written for the vectors of the superdifferential as follows:

$$-X_1 - X_2 - a + |2\xi_1 - 2X_2 - a| + |2\xi_2| \ge 0, \quad a \in [0, \infty] \tag{5.7}$$

Clearly, this inequality holds, since the expression in the first term with absolute value when expanded has a negative sign, while the second is positive.

The subdifferential of the function Φ on the surface BC is the empty set, and so inequality (5.3) holds automatically.

We will now verify that the differential inequalities for the generalized gradients of the value function also hold on the edge of the controllability cone – the line B.

The directional derivative of the value function on the line B is

$$\partial\Phi_e(T, T_0, X) = \begin{cases} -\infty \quad \text{if} \quad e_{T_0} + e_{X_1} > 0, \quad e_{T_0} + e_{X_2} > 0 \\ \langle \nabla F_1, e \rangle \quad \text{otherwise} \end{cases}$$

(the function F_1 is defined by the first formula of (5.2)). For the superdifferential we obtain a formula that differs from (5.6) only in the presence of the displacement vector $(-b, 0, -b)$, defined by the additional degree of freedom b, $b \in [0, \infty]$, that is, the superdifferential is a convex cone with apex at the point ∇F_1, generated by the vectors $(-1, -1, 0)$, $(-1, 0, -1)$. Then inequality (5.4) for the vectors of the superdifferential differs from (5.7) in that the third term on the right is replaced by $-(a + b)$ and the last term by $|2\xi_2 - b|$. It holds for the same reasons as (5.7).

Thus, we have verified all Subbotin's differential inequalities for the value function W, that is, we have verified that it satisfies the necessary and sufficient optimality conditions and is a generalized solution of the Hamilton–Jacobi equation.

We wish to dedicate this paper to the eightieth birthday of Academician N. N. Krasovskii.

This research was supported financially by the Russian Foundation for Basic Research (02-01-00769) and the "Government Support for Leading Scientific Schools" programme (NSh-791.2003.1).

REFERENCES

1. KRASOVSKII, A. N. and KRASOVSKII, N. N., *Control under Lack of Information*. Birkhauser, Basel, 1994.
2. PONTRYAGIN, L. S., BOLTYANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, Ye. F., *Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1976.
3. BROCKETT, R. W., Asymptotic stability and feedback stabilization. In *Differential Geometric Control Theory* (edited by R. W. Brockett *et al.*). Birkhauser, Boston, 1983, 181–191.
4. ASTOLFI, A. and RAPAPORT, A., Robust stabilization of the angular velocity of a rigid body. *Systems and Control Letters*, 1998, **34**, 257–264.
5. CLARKE, F. H., LEDYAYEV, Yu. S. and STERN, R. J., Proximal analysis and feedback construction. *Trudy Inst. Mat. Mekh., Ural. Otd. Ross. Akad. Nauk*, 2000, **6**, 1, 91–109.
6. NIKOL'SLKII, M. S., Interior estimation of the attainability set of a non-linear Brockett integrator. *Diff. Uravn*., 2000, **96**, 11, 1501–1505.
7. CHERNOUS'KO, F. L. and MELIKYAN, A. A., *Game Problems of Control and Search*. Nauka, Moscow, 1978.
8. CHERNOUS'KO, F. L., BOLOTNIK, N. N. and GRADETSKII, V. G., *Manipulator Robots: Dynamics, Control, Optimization*. Nauka, Moscow, 1989.
9. BLAGODATSKIKH, V. I., *Introduction to Optimal Control (Linear Theory)*. Vysshaya Shkola, Moscow, 2001.
10. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
11. SUBBOTIN, A. I., *Minimax Inequalities and the Hamilton–Jacobi Equations*. Nauka, Moscow, 1991.
12. SUBBOTIN, A. I. and CHENTSOV, A. G., *Optimization of Guarantee in Control Problems*. Nauka, Moscow, 1981.
13. SUBBOTIN, A. I., *Generalized Solutions of First-Order PDEs: The Dynamical Optimization Perspective*. Birkhauser, Boston, 1995.
14. SUBBOTIN, A. I., The method of Cauchy characteristics and generalized solutions of the Hamilton–Jacobi-Bellman equations. *Dokl. Akad. Nauk SSSR*, 1991, **320**, 3, 556–561.
15. SUBBOTIN, A. I., TARAS'VEV, A. M and USHAKOV, V. N., Generalized characteristics of the Hamilton–Jacobi equations. *Izv. Akad. Nauk. Tekhn. Kibernetika*, 1993, 1, 190–197.
16. KARTASHEV, A. P. and ROZHDESTVENSKII, B. L., *Ordinary Differential Equations and the Elements of the Variational Calculus*. Nauka, Moscow, 1976.

Translated by D.L.